An Algebraic Investigation of Constrained and Bandaged Rubik's Cubes

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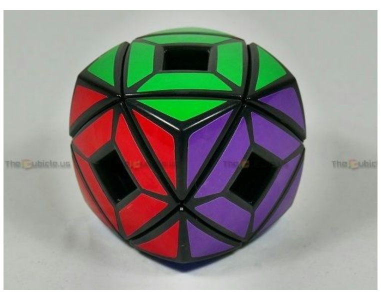
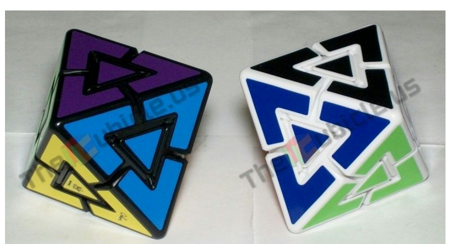
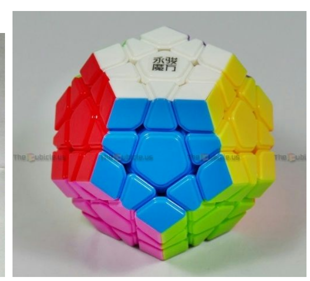
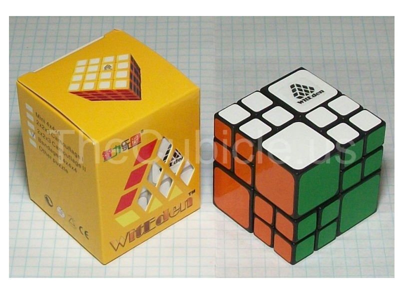
10-week project to begin May 31

Since its invention in 1974, the Rubik's cube has been a source of endless discovery and competition. Speed-solvers learn algorithms and practice techniques to find the fastest time to solve the cube. Engineers have developed robots that can solve the cube even faster [1]. It has been applied to cryptography ([2] and [3]). Mathematicians have modeled the possible moves of the cube using algebraic structures and used that model to address deeper questions ([4] and [5]). One of the more recent results proved about the cube was finding "God's Number" — the smallest number of moves necessary to solve *any* scrambled cube (in mathematical terms, this is the diameter of the associated Cayley graph). Puzzlers long speculated that God's Number was 20, and this result was finally proven in 2010 [6], with the result summarized in 2014 in a special issue of the *College Mathematics Journal* dedicated to the Rubik's cube [7].

Outstanding questions still exist for the cube. One example is the search for a "Devil's Algorithm" — a set of moves that is guaranteed to solve the cube regardless of the starting position (normally, you have to study the scrambled cube first to figure out which algorithms to apply). Over the decades, scores of variations of the cube have been developed. There are 4x4x4, 5x5x5, and even larger cubes. Puzzles have been created based on just about every platonic solid. Below are just a few variations currently on the market (images from http://thecubicle.us/).

Some of these variations have been subject to the same mathematical analysis as the original Rubik's cube. Joyner's text [3] provides an excellent introduction to group theory using several variations of these puzzles. But some of the puzzles have not yet been analyzed and provide special challenges in developing a group-theoretical model.



For our summer project, we plan to focus on two variations of the Rubik's cube that still pose open challenges: the bandaged cube and the constrained cube. The bandaged cube is a cube in which some of the adjacent "cubies" (the small colored blocks that make up the full cube) are literally bandaged together. This can be done using strong tape on a standard cube or by purchasing specially-made bandaged cubes (image from http://thecubicle.us/).

Constrained cubes have restrictions on how much you can turn any given face — for the constrained cubes we will study, each face only has a 90° range of motion, rather than the full 360° that you can normally turn the faces of a Rubik's cube.

Our plan is to develop an algebraic model for some small bandaged and constrained cubes. With that model, we hope to answer some basic open questions about these variations:

- How many moves/configurations are possible for a given variation?
- Can we tell which positions are "legal"? That is, if someone took the cube apart and then put it back together in a scrambled state, can the scrambled cube be solved short of taking it apart again?
- What is "God's Number" for these variations?

Roles and responsibilities

Our work will be joint mathematical analysis. The role of the faculty will be to help with the algebraic background needed for the analysis, and the student will play a central role in building the algebraic model and developing the relevant theorems and proofs uncovered through our work, as roughly outlined here.

As a first-year freshman, Bryan has already shown himself to have very strong mathematical skills, a tremendous academic curiosity, and a strong ability to work on complex problems. He has already begun doing independent reading on the algebra background needed for this project. He will be an active participant in new mathematical research, and our final written work will be highly collaborative.

Timetable and milestones

- **Week 1**: Study algebraic model for standard Rubik's cube. This will constitute the background work needed to review current mathematical understanding of the cube.
- **Weeks 2 and 3**: Develop and begin to analyze the group structure of the 3x3 bandaged cube. This group model should be fairly straightforward to develop, and the analysis should be an extrapolation of the work on the classic Rubik's cube.
- **Weeks 4 through 8**: Develop a model for a 4x4 bandaged cube and/or the 3x3 constrained cube. These models will likely be much more challenging to develop. During this time, we will also continue our analysis of the 3x3 bandaged cube — partly to finish our work from weeks 2-3, and partly to help understand how the model will generalize to the more complex cube variants.
- **Weeks 9 and 10**: Wrap up our work, write a summary of our results, and take stock of open questions that remain.

Budget

While some minor computational work will need to be done — which can easily be accomplished using our existing laptops and software — most of the work will be pen-and-paper activity. As such we have no budget requests beyond the stipend and housing allowances.

Anticipated Outcomes/Contribution to the Discipline

Bryan will be conducting pure mathematical research. This will be new research for which Bryan has a hand in developing the questions, finding/proving results, and creating a written article to be shared with the mathematical community. At a minimum, he will be able to submit his results to an undergraduate mathematical publication, such as the *Rose-Hulman Undergraduate Mathematics Journal*. Our hope is that our joint work will be publishable in a venue such as *PRIMUS* or the *College Mathematics Journal*. In any case, Bryan will be author or co-author of the publication.

In addition, there will be multiple opportunities for Brian to present the work from this SOAR project to the mathematical community, including the Moravian Student Mathematics Conference and the regional EPaDel Mathematics Conferences hosted by the MAA.

Citations

1. Robot Solves Rubik's Cube in 5 Seconds, http://www.zdnet.com/article/robot-solves-rubiks-cube-in-5-seconds-sets-world-record-video/
2. "An improved secure image encryption algorithm based on Rubik's cube principle and digital chaotic cipher," by Adrian-Viorel Diaconu and Khaled Loukhaoukha, *Mathematical Problems in Engineering*, 2013
3. "Rubik's for cryptographers," by Christophe Petit and Jean-Jacques Quisquater, *Notices of the AMS*, vol 60, no 6, 2013, 733–740.
4. "On the continual Rubik's cube," by Dmitry Ryabogin, Advances in Mathematics, vol 231, no 6, 2012, 3429–3444.
5. "Rubik's on the torus," Jeremy Alm, Michael Gramelspacher, and Theodore Rice, *American Mathematical Monthly*, vol 120, no 2 (2013), 150–160.
6. "The diameter of the Rubik's cube group is twenty," Tomas Rokicki, Herbert Kociemba, Morley Davidson, and John Dethridge, *SIAM Journal of Discrete Math*, vol 27, no 2, 2013, 1082–1105.
7. "Towards God's Number for Rubik's Cube in the Quarter-Turn Metric", by Tomas Rokicki, *College Mathematics Journal*, vol 45, no 4, September 2014
8. *Adventures in Group Theory: Rubik's Cube, Merlin's Machine, and Other Mathematical Toys*, by David Joyner, JHU Press, 2008

Student Statement of Purpose

An Algebraic Investigation of Constrained and Bandaged Rubik's Cubes
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On campus housing is requested

I have been solving different kinds of Rubik's Cubes since I was in 6th grade, and have developed a more mathematical interest in them recently, particularly in metrics that determine how difficult a twisty puzzle is. Most of my efforts up to very recently have been focused on solving different kinds of Rubik's Cube variations, like mix-up puzzles, some bandaging puzzles and wormhole cubes. I also have done a fair amount of speed solving on most WCA puzzles over the past year and a half or so. Now that I have a stronger mathematical background, I am extremely interested in studying a much more mathematical approach to analyzing different puzzles.

Taking Moravian's Discrete Mathematics course (MATH 216), I have learned some Group Theory and counting concepts and am currently studying how they apply to Rubik's Cubes on the side. Part of my interest in this project is wrapped up in my experience with trying to solve different puzzles algorithmically without having a strong understanding of what is happening mathematically. Trying to solve a puzzle though is much different from analyzing it algebraically. For instance, knowing what an algorithm will do, does not explain why it works. One of the things that interests me most about constrained and bandaged cubes is what permutations are possible after restricting movement, and why other permutations are not possible.

Before the SOAR project would even begin I will be reading some texts and papers to learn key algebraic concepts that were not covered in MATH 216.This will allow me to finish up learning the relevant algebra concepts by the end of the first week of the project. By the end of the summer I hope to have working algebraic models for constrained cubes, as well as a few different bandaging puzzles. For the constrained cube series, there are 4 different puzzles to potentially model, but currently we are planning on focusing on the constrained (90°). For the bandaged puzzles, I expect that we would have a functioning algebraic model for at least the "Fused Cube" and a few other similar ones. If the project is proceeding well, I would like to figure out how to model bandaged cubes with multiple bandages or bandaging on larger and/or even order cubes.

Another thing that intrigues me about these puzzles are the different measurements of difficulty. One simple measure of difficulty which is usually calculable is the number of possible permutations of a puzzle. For example, on a 3x3x3 Rubik's Cube, there are roughly 4.3×10^{19} permutations, and on a 2x2x2 'pocket cube' there are roughly 3.6×10^{6} permutations. The other, much more difficult measure of a puzzles difficulty is the God's Number for a puzzle. For a 3x3x3, the God's Number is 20, and for a 2x2x2, it is 11. With bandaging cubes, I expect that we will see that the number of permutations will go down quite a bit, and with Constrained Cubes I expect that both number of permutations and God's Number will increase. I hope to be able to find at least some general bounds for the God's Number of each cube.

With this information I intend to author/co-author an article with the intentions of getting it published in at least one mathematics journal. I plan to continue math and Physics research throughout my undergraduate years, into my graduate school, and this SOAR project would be a great chance for me to learn how to conduct a research project, which will prepare me for REU projects at other colleges in the future.

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